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Supertask Computation

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Abstract. Infinite time Turing machines extend the classical Turing machine concept to transfinite ordinal time, thereby providing a natural model of infinitary computability that sheds light on the power and limitations of supertask algorithms.

1 Supertasks

What would you compute with an infinitely fast compute? What *could* you compute? To make sense of these questions, one would want to understand the algorithms that the machines would carry out, computational tasks involving infinitely many steps of computation. Such tasks, known as *supertasks*, have been studied since antiquity from a variety of viewpoints.

Zeno of Elea (ca. 450 B.C.) was perhaps the first to grapple with supertasks, in his famous paradox that it is impossible to go from here

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to there, because before doing so one must first get halfway there, and before that halfway to the halfway point, and so on, *ad infinitum*. Zeno takes the impossibility of completing a supertask as the foundation of his reductio. More recently, twentieth century philosophers (see [Tho54]) have introduced Thomson's lamp, which is on for 1/2 minute, off for 1/4 minute, on for 1/8 minute, and so on. After one minute, is it on or off?

In a more intriguing example, let's suppose that you have infinitely many one dollar bills (numbered 1, 3, 5, \cdots) and in some nefarious underground bar, the Devil explains to you that he has an attachment to your particular bills, and is willing to pay you two dollars for each of your one dollar bills. To carry out the exchange, he proposes an infinite series of transactions, in each of which he will hand over to you two dollars and take from you one dollar. The first transaction will take 1/2 hour, the second 1/4 hour, the third 1/8 hour, and so on, so that after one hour the entire exchange is complete. Should you accept his proposal? Perhaps you will become richer? At the very least, you think, it will do no harm, and so the contract is signed and the procedure begins.

It appears initially that you have made a good bargain, because at every step of the transaction, you receive two dollars but give up only one. The Devil is particular, however, about the order in which the bills are exchanged: he always buys from you your lowest-numbered bill, paying you with higher-numbered bills. (So on the first transaction he accepts from you bill number 1, and pays you with bills numbered 2 and 4, and on the second transaction he buys from you bill number 2, which he had just paid you, and pays you bills numbered 6 and 8, and so on.) When the transaction is complete, you discover that you have no money left at all! The reason is that at the n^{th} exchange, the Devil took from you bill number n, and never subsequently returned it to you. Thus, the final destination of every individual bill is under the ownership of that shrewd banker, the Devil.

The point is that you should have paid more attention to the details of the supertask transaction that you had agreed to undertake. And similarly, when we design supertask algorithms to solve mathematical questions, we must take care not to make inadvertent assumptions about what may be true only for finite algorithms.

Supertasks have also been studied by the physicists (see [Ear95]). Using only the Newtonian gravity law (and neglecting relativity), it is possible to arrange finitely many stars in orbiting pairs, each pair orbiting the common center of mass of all the pairs, and a single tiny moon racing faster around, squeezing just so between the dual stars so as to pick up speed with every such transaction. Assuming point masses (or collapsing stars to avoid collision), the arrangement leads by Newton's law of gravitation to infinite acceleration in finite time. Other supertasks reveal apparent violations of the conservation of energy in Newtonian physics: infinitely many billiard balls, of successively diminishing size converging to a point, are initially at rest, but then the first is set rolling, and each ball transfers in turn all the energy to the next; after a finite amount of time, all motion has ceased, though every interaction is energy conserving. Still other arrangements have the balls spaced out further and further out to infinity, and the interesting thing about both of these examples is that time-symmetry allows them to run in reverse, with static configurations of balls suddenly coming into motion without violating conservation of energy in any interaction.

More computationally significant supertasks have been proposed by physicists in the context of relativity theory ([EN93], [Hog92], [Hog94]). Suppose that you want to know the answer to some number theoretic conjecture, such as whether there are additional Fermat primes (primes of the form $2^{2^n} + 1$), a conjecture that can be confirmed with a single numerical example. The way to solve the problem is to board a rocket, while setting your graduate students to work on earth looking for an example. While you fly faster and faster around the earth, your graduate students, and their graduate students and so on, continue the exhaustive search, with the agreement that if they ever find an example, they will send a radio signal up to the rocket. The point is that meanwhile, by accelerating sufficiently fast towards the speed of light, it is possible to arrange that because of relativistic time contraction, what is a finite amount of time on the rocket corresponds to an infinite amount of time on the earth. The general observation is that by means of such communication between two reference frames, what corresponds to an infinite search can be completed in a finite amount of time.

Even more complicated arrangements, with rockets flying around rockets, can be arranged to solve more complicated number theoretic questions. And more complicated relativistic spacetimes can be (mathematically) constructed to avoid the unpleasantness of infinite acceleration required in the rocket examples above (see [Pit90]).

These computational examples speak to Church's thesis, the widely accepted philosophical principle that the classical theory of computability has correctly captured the notion of what it means to be computable. Because the relativistic rocket examples provide algorithms for computing functions, such as the halting problem, that are not computable by Turing machines, one can view them as refuting Church's thesis. Supporters of this view emphasize that when thinking about what is in principle computable, we must attend to the computational power available to us as a consequence of the fact that we live in a relativistic or quantummechanical universe. To ignore this power is to pretend that we live in a Newtonian world. Another simpler argument against Church's thesis consists of the observation that a particle undergoing Brownian motion can be used to generate a random bit stream that we have no reason to think is recursive. Therefore, proponents argue, we have no reason to believe Church's thesis.

Apart from the question of what one can actually compute in this world, whether Newtonian or relativistic or quantum-mechanical, mathematicians are interested in what in principle a supertask can accomplish. Buchi [Buc62] and others initiated the study of ω -automata and Buchi machines, involving automata and Turing machine computations of length ω which accept or reject infinite input. Moving to a higher level in the hierarchy, Gerald Sacks and many others (see [Sac90]) founded the field of higher recursion theory, including α -recursion and E-recursion, a huge body of work analyzing computation on infinite objects. Blum, Shub and Smale [BSS89] have presented a model of computation on the real numbers, a kind of flowchart machine where the basic units of computation consist of real numbers, in full glorious precision. Apart from this previous mathematical work, I would like to propose here a new model of infinitary computability: infinite time Turing machines. This model offers the strong computational power of higher recursion theory while remaining very close in spirit to the computability concept of ordinary Turing machines.

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2 Infinite time Turing machines

I propose to extend the Turing machine concept to transfinite ordinal time, thereby providing a natural model for infinitary computability.¹ The idea is to allow somehow a Turing machine to compute for infinitely many steps, while preserving the information produced up to that point.

So let me explain specifically how the machines work. The machine hardware is identical to a classical Turing machine, with a head moving back and forth reading and writing zeros and ones on a tape according to the rigid instructions of a finite program, with finitely many states. What is new is the transfinite behavior of the machine, behavior providing a natural theory of computation on the reals that directly generalizes the classical finite theory to the transfinite. For convenience, the machines



Fig. 1. An infinite time Turing machine: the computation begins

have three tapes—one for the input, one for scratch work and one for the output—and the computation begins with the input written out on the input tape, with the head on the left-most cell in the *start* state. The successor steps of computation proceed in exactly the classical manner: the head reads the contents of the cells on which it rests, reflects on its state and follows the rigid instructions of the finite program it is running: accordingly, it writes on the tape, moves the head one cell to the left or the right or not at all and switches to a new state. Thus, the classical procedure determines the configuration of the machine at stage $\alpha + 1$, given the configuration at any stage α .

¹ Infinite time Turing machines were originally defined by Jeff Kidder in 1990, and he and I worked out the early theory together while we were graduate students at UC Berkeley. Later, Andy Lewis and I solved some of the early questions, and presented a complete introduction in [HL00], later solving Post's problem for supertasks in [HL]. Benedikt Loewe [Low01], Dan Seabold [HS01] and especially Philip Welch [Wel99], [Wel00], [Wel] have also made important contributions.

We extend the computation into transfinite ordinal time by simply specifying the behavior of the machine at limit ordinals. When a classical Turing machine fails to halt, it is usually thought of as some sort of failure; the result is discarded even though the machine might have been writing some very interesting information on the tape (such as all the theorems of mathematics, for example, or the members of some other computably enumerable set). With infinite time Turing machines, however, we hope to preserve this information by taking some kind of limit of the earlier configurations and continuing the computation transfinitely. Specifically, at any limit ordinal stage λ , the head resets to the left-most cell; the machine is placed in the special *limit* state, just another of the finitely many states; and the values in the cells of the tape are updated by computing the lim sup of the previous cell values. With the limit stage

	limit	•					
input:	1	1	0	1	0	0	
scratch:	0	1	1	0	0	1	
output:	1	1	0	1	1	1	

Fig. 2. The limit configuration

configuration thus completely specified, the machine simply continues computing. If after some amount of time the *halt* state is reached, the machine gives as output whatever is written on the output tape.

Because there seems to be no need to limit ourselves to finite input and output—the machines have plenty of time to consult the entire input tape and to write on the entire output tape before halting—the natural context for these machines is Cantor Space 2^{ω} , the space of infinite binary sequences. For our purposes here, let's denote this space by \mathbb{R} and refer to its members as real numbers, intending by this terminology to mean infinite binary sequences. We regard the set of natural numbers \mathbb{N} as a subset of \mathbb{R} by identifying the number 0 with the sequence $\langle 000 \cdots \rangle$, the number 1 with $\langle 100 \cdots \rangle$, the number 2 with $\langle 110 \cdots \rangle$, and so on.

Because every program p determines a function—the function sending input x to the output of the computation of program p on input x—the machines provide a model of computation on the reals. We define that a

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partial function $f : \mathbb{R} \to \mathbb{R}$ is *infinite time computable* (or *supertask computable*, or for brevity, just *computable*, when the infinite time context is understood) when there is a program p such that f(x) = y if and only if the computation of program p on input x yields output y. A set of reals $A \subseteq \mathbb{R}$ is *infinite time decidable* (or *supertask decidable* or again, just *decidable*) when its characteristic function, the function with value 1 for inputs in A and 0 for inputs not in A, is computable. The set A is infinite time *semi-decidable* when the function of affirmative values $1 \upharpoonright A$, that is, the function with domain A and constant value 1, is computable. (Thus, the semi-decidable sets correspond in the classical theory to the recursively enumerable sets, though since here we have sets of reals, we hesitate to describe them as enumerable.) Since it is an easy matter to change any output value to 1, the semi-decidable sets are exactly the domains of the computable functions, just as in the classical theory.

Theorem 1. Every supertask computation halts or repeats in countably many steps.

Proof. Suppose that a supertask computation does not halt by any countable stage of computation. The point is now that a simple cofinality argument shows that the complete configuration of the machine at stage ω_1 the position of the head, the state and the contents of the cells-must have occurred earlier. For example, one can find a countable ordinal α_0 by which time all of the cells that have stabilized by ω_1 have stabilized. And then one can construct a countable increasing sequence of countable ordinals $\alpha_0 < \alpha_1 < \cdots$ such that all the cells that change their value after α_n do so at least once between α_n and α_{n+1} . These ordinals exist because ω_1 is regular and there are only countably many cells. At the limit stage $\alpha_{\omega} = \sup \alpha_n$, which is still a countable ordinal, I claim that the configuration is the same as at ω_1 : since it is a limit ordinal, the head is on the first cell and in the limit state; and by construction the contents of each cell are computing the same lim sup that they compute at ω_1 . Since beyond α_0 the only cells that change are the ones that will change unboundedly often, it follows that limits of this configuration are the very same configuration again, and the machine is caught in an endlessly repeating loop. So the proof is complete.

Please observe in this argument that, contrary to the classical situation, a computation that merely repeats a complete machine configuration need not be caught in an endlessly repeating loop. After ω many repetitions, the limit configuration may allow it to escape. One example of this phenomenon would be the machine which does nothing at all except halt when it is in the *limit* state; this machine repeats its initial configuration many times, yet still halts at ω .

3 How powerful are the machines?

One naturally wants to understand the power of the new machines. The first observation, of course, is that the classical halting problem for ordinary Turing machines—the question of whether a given program p halts on given input n in finitely many steps—is decidable in ω many steps by an infinite time Turing machine. To see this, one programs an infinite time Turing machine to simply simulate the operation of p on n, and if the simulated computation ever halts our algorithm gives the output that yes, indeed, the computation did halt. Otherwise, the *limit* state will be attained, and when this occurs the machine will that know the simulated computation failed to halt; so it outputs the answer that no, the computation did not halt.

The power of infinite time Turing machines, though, far transcends the classical halting problem. The truth is that any question of first order number theory is supertask decidable. With an infinite time Turing machine, one could solve the prime pairs conjecture (which asserts that there are infinitely many primes pairs, pairs of primes differing by two), for example, and the question of whether there are infinitely many Fermat primes (primes of the form $2^{2^n} + 1$) and so on: there is a general decision algorithm for any such conjecture. The point is that to decide a question of the form $\exists n \varphi(n, x)$, where n ranges over the natural numbers, one can simply try out all the possible values of n in turn. One either finds a witness n or else knows at the limit that there is no such witness, and in this way decides whether $\exists n \varphi(n, x)$. Iterating this idea, one concludes by induction on the complexity of the statement that any first order number theoretic question is decidable with only a finite number of limits, that is, before stage ω^2 . In fact, the class of sets that are decidable in time uniformly before ω^2 is exactly the class of arithmetic sets, the sets of reals that are definable by a statement using quantifiers over the natural numbers (see [HL00, Theorem 2.6]).

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Theorem 2. Arithmetic truth is infinite time decidable.

One can push this much harder to see that even more complex questions, questions from the lower part of the projective hierarchy in second order number theory, are supertask decidable. The fact is that any Π_1^1 set is decidable and more. To prove this, it suffices to consider the most complex Π_1^1 set, the well-known set WO, consisting of the reals coding a well-orders of a subset of \mathbb{N} . An infinite binary sequence x codes a relation \triangleleft on \mathbb{N} when $i \triangleleft j$ if and only if $x(\langle i, j \rangle) = 1$, where $\langle \cdot, \cdot \rangle$ is the Gödel pairing function coding pairs of natural numbers with natural numbers.

Theorem 3. *The set* WO *is infinite time decidable.*

Proof. This argument is known as the "count-through" argument. We would like to describe a supertask algorithm which on input x decides whether x codes a well order \lhd on a subset of \mathbb{N} or not. In ω many steps, it is easy to check whether x codes a linear order: this amounts merely to checking that the relation \lhd coded by x is transitive, irreflexive and connected. For example, the machine must check that whenever $i \lhd j$ and $j \lhd k$ then also $i \lhd k$, and all these requirements can be enumerated and checked in ω many steps.

Next, the algorithm will attempt to find the least element in the field of the relation \triangleleft . This can be done by keeping a current-best-guess on the scratch tape and systematically looking for better guesses, whenever a new smaller element is found. When such a better guess is found, it replaces the current guess on the scratch tape, and a special flag cell is flashed on and then off again. At the limit, if the flag is on, it means that infinitely often the guess was changed, and so the relation has an infinite descending sequence. Thus, in this case the input is definitely not a well order and the computation can halt with a negative output. Conversely, if the flag is off, it means that the guess was only changed finitely often, and the machine has successfully found the \triangleleft least element. The algorithm now proceeds to erase all mention of this element from the field of the relation \triangleleft . This produces a new smaller relation, and the algorithm proceeds to find the least element of it. In this way, the relation \triangleleft is eventually erased from the bottom as the computation proceeds. If the relation is not a well order, eventually the algorithm will erase the well founded initial segment of it, and then discover that there is no least element remaining, and reject the input. If the relation is a well order,

then the algorithm will eventually erase the entire field, and recognize that it has done so, and accept the input as a well order. This completes the proof.

Since W0 is well-known as a complete Π_1^1 set, we conclude as a corollary that every Π_1^1 set is infinite time decidable and hence also, every Σ_1^1 set is infinite time decidable. But one can't go much further in the projective hierarchy, because every semi-decidable set has complexity Δ_2^1 . For a finer stratification, let me mention that the arithmetic sets are exactly the sets which can be decided by an algorithm using a bounded finite number of limits, and the hyperarithmetic sets, the Δ_1^1 sets, are exactly the sets which can be decided in some bounded recursive ordinal length of time. Thus, the arithmetic sets are those that can be decided uniformly in time before ω^2 , and the hyperarithmetic sets are exactly those which can be decided uniformly in time before ω_1^{ck} .

Much of the classical computability theory generalizes to the supertask context of infinite time Turing machines. For example, the s-m-n theorem and the Recursion Theorem go through with virtually identical proofs. But some other classical results, even very elementary ones, do not generalize. One surprising result, for example, is the following.

Theorem 4. *There is a non-computable function whose graph is semi-decidable.*

This follows from what I have called the Lost Melody Theorem [HL00, Theorem 4.9], which asserts the existence of a real c such that $\{c\}$ is decidable, but c is not writable. Imagine the real c as the melody that you can recognize when someone sings it, but you cannot sing it on your own. Using such a lost melody real c, one can prove Theorem 4 with the function f(x) = c. Indeed, since this function is constant and the graph is decidable, the theorem can be strengthened to the assertion that there is a non-computable constant function whose graph is decidable. To give some idea of how one proves the Lost Melody Theorem, let me mention that the real c will be the least real in the Gödel constructible universe L hierarchy that codes the ordinal supremum of the places where all computations on input 0 have either halted or repeated. Since this ordinal is above every writable ordinal, the real c cannot be writable. But the real c codes enough information about itself so that an infinite time Turing machine can verify that a given real is c or not.

4 How long do the computations take?

One naturally wants to understand how long a supertask computation can take. Therefore, I define an ordinal α to be *clockable* if there is a computation on input 0 that takes exactly α many steps to complete (so that the α^{th} step of computation is the act of moving to the halt state). Such a computation is a clock of sorts, a way to count exactly up to α .

It is very easy to see that any finite n is clockable; one can simply have a machine cycle through n states and then halt. The ordinal ω is clockable, by the machine that halts whenever it sees the *limit* state. And these same ideas show that if α is clockable, then so is $\alpha + n$ and $\alpha + \omega$. Thus, every ordinal up to ω^2 is clockable. The ordinal ω^2 itself is clockable: one can recognize it as the first limit of limit ordinals, by flashing a flag on and then off again every time the *limit* state is encountered. The ordinal ω^2 will be first time this flag is on at a limit stage. Going beyond this, it is easy to see that if α and β are clockable, so are $\alpha + \beta$ and $\alpha\beta$. Undergraduate students might enjoy finding algorithms to clock specific ordinals, such as ω^{ω^2} , and I can recommend this as a way to help them understand the ordinals more deeply.

Most readers will have guessed that the analysis extends much further. In fact, any recursive ordinal is clockable. This can be seen by optimizing the count-through argument in Theorem 3. Specifically, after writing a real coding a recursive ordinal on the tape in ω many steps, one proceeds to count through it in an optimized fashion. Rather than merely guessing the least element of the relation, one guesses the ω many least elements of the relation (while simultaneously erasing the previous guesses). In this way, each block of ω many steps of the algorithm will erase ω many elements from the field of the relation.

Some have been surprised that the clockable ordinals extend beyond the recursive ordinals, but in fact they extend well beyond the recursive ordinals. To see at least the beginnings of this, let me show that the ordinal $\omega_1^{ck} + \omega$ is clockable, where ω_1^{ck} is the supremum of the recursive ordinals. Kleene has proved that there is a recursive relation whose wellfounded part has order type ω_1^{ck} . Consider the supertask algorithm that writes this relation on the tape and then attempts to count through it. By stage ω_1^{ck} the ill-founded part will have been reached, but it takes the algorithm an additional ω many steps to realize this. So it can halt at stage $\omega_1^{ck} + \omega$.

One is left to wonder, is ω_1^{ck} itself clockable? More generally, *Are there gaps in the clockable ordinals*? After all, if a child can count to twenty-seven, then one might expect the child also to be able to count to any smaller number, such as nineteen.² The question is whether we expect the same to be true for infinite time Turing machines.

Theorem 5. *Gaps exist in the clockable ordinals.*

Proof. Consider the algorithm which simulates all programs on input 0, recording which have halted. When a stage is found at which no programs halt, then halt. This produces a clockable ordinal above a non-clockable ordinal, so gaps exist.

The argument can be modified to show that the next gap above any clockable ordinal has size ω . Other arguments establish that complicated behavior can occur at limits of gaps, because the lengths of the gaps are unbounded in the clockable ordinals.

Question 1. What is the structure of the clockable ordinals?

For example, one might wonder whether the first gap begins at ω_1^{ck} , the supremum of the recursive ordinals? (It does, since no admissible ordinal is clockable [HL00].)

There is another way for infinite time Turing machines to operate as clocks, and this is by counting through a real coding a well order in the manner of Theorem 3. To assist with this analysis, we define that a real is *writable* if it is the output of a supertask computation on input 0. An ordinal is writable if it is coded by a writable real. It is easy to see that there are no gaps in the writable ordinals, because if one can write down real coding α , it is an easy matter to write down from this a real coding any particular $\beta < \alpha$. In [HL00], Andy Lewis and I proved that the order types of the clockable and writable ordinals are the same, but the question was left open as to whether these two classes of ordinals had the same supremum. This was solved by Philip Welch in [Wel], allowing Andy Lewis and I to greatly simplify arguments in [HL].

² Friends with children have informed me that such an expectation is unwarranted; one sometimes can't get the child to stop at the right time. This reminds me of a time when my younger brother was in kindergarten, the children all sat in a big circle taking turns saying the next letter of the alphabet: A, B, C, and so on, around the circle in the manner of the usual song. After the letter K, the next child contributed LMNOP, thinking that this was only one letter.

Theorem 6. (Welch) *Every clockable ordinal is writable. The supremum of the writable and clockable ordinals is the same.*

5 The supertask halting problems

Any notion of computation naturally provides a corresponding halting problem, the question of whether a given computation will halt. In the supertask context, we divide the halting problem into two parts, a boldface and a lightface problem:

 $H = \{ \langle p, x \rangle \mid \text{program } p \text{ halts on input } x \}$ $h = \{ p \mid \text{program } p \text{ halts on input } 0 \}$

In the classical theory, of course, these two sets are Turing equivalent, but here the situation is different. Nevertheless, for undecidability the classical arguments do directly generalize.

Theorem 7. The halting problems h and H are semi-decidable but not decidable.

For semi-decidability, the point is that given a program p and input x (or input 0), one can simply simulate p on x to see if it halts. If it does, output the answer that yes, it halted; otherwise, keep simulating. For undecidability, in the case of H one can use the classical diagonalization argument; for the lightface halting problem h, one appeals to the Recursion Theorem, just as in the classical theory.

6 Oracles

There are two natural types of oracles to use in the infinite time Turing machine context. On the one hand, one can use an individual real as an oracle just as one does in the classical context, by simply adding an oracle tape containing this real, and allowing the machine to access this tape during the computation. This corresponds exactly to adding an extra input tape and thinking of the oracle real as a fixed additional input.

But this is ultimately not the right type of oracle to consider. Rather, an oracle is more properly the same type of object as one that might be decidable or semi-decidable, namely, a *set* of reals, not an individual real. Since such a set could be uncountable, we can't expect to be able to write out the entire contents of the oracle on an extra tape. Rather, we provide an oracle model of relative computability by which the machine can make arbitrary membership queries of the oracle. Specifically, for a fixed oracle set of reals A, we equip an infinite time Turing machine with an initially blank oracle tape on which the machine can read or write. By attempting to switch to a special *query* state, the machine receives the answer (by moving actually to the *yes* or *no* state) as to whether the real currently written on the oracle tape is in A or not. In this way, the machine is able to ask, of any real x that it is capable of producing, whether $x \in A$ or not. This model of oracle computation has proven robust, and it closely follows the well-known definition of L[A] in set theory, the constructible universe relative to the predicate A, in which at any given stage in the construction one is allowed to apply the predicate only to previously constructed objects.

From the notion of oracle computation, one can of course define a notion of relative computability. Specifically, the set A is computable from B, written $A \leq_{\infty} B$, if and only if A is supertask decidable using oracle B. One then also defines $A \equiv_{\infty} B$ if and only if $A \leq_{\infty} B$ and $B \leq_{\infty} A$, and this is the equivalence relation of the infinite time Turing degrees. The strict version $A <_{\infty} B$ holds if and only if $A \leq_{\infty} B$ and $A \not\equiv_{\infty} B$.

7 Supertask Jump Operators

The two halting problems give rise of course to two jump operators. Specifically, for any set A we have the boldface and lightface jumps:

 $A^{\mathbf{V}} = H^A = \{ \langle p, x \rangle \mid \text{program } p \text{ halts on input } x \text{ with oracle } A \}$

 $A^{\nabla} = A \oplus h^A = A \oplus \{ p \mid \text{program } p \text{ halts on input } 0 \text{ with oracle } A \}$

We include the factor A explicitly in A^{∇} , because in general A may not be computable from h^A . Indeed, there are some sets A that are not computable from any real at all.

Jump Theorem 8 For any set, $A <_{\infty} A^{\nabla} <_{\infty} A^{\nabla}$.

To prove this theorem, one first observes that $A \leq_{\infty} A^{\nabla} \leq_{\infty} A^{\nabla}$, since A is explicitly computable from A^{∇} and A^{∇} is merely the 0^{th} slice of

 A^{∇} . Secondly, one knows that $A <_{\infty} A^{\nabla}$ because the undecidability of the relativized halting problem means that h^A is not computable from A. The nontrivial aspect of this theorem is the assertion that $A^{\nabla} <_{\infty} A^{\nabla}$. This assertion is what separates the two jump operators, and is the reason that we know the two halting problems $h \equiv_{\infty} 0^{\nabla}$ and $H \equiv_{\infty} 0^{\nabla}$ are not equivalent. This follows from the more specific result that the set A^{∇} is not computable from $A \oplus z$ for any real z. In particular, 0^{∇} is not computable from any real. In fact the boldface jump ∇ jumps much higher than the lightface jump ∇ , and absorbs many iterates of the weaker jump, since $A^{\nabla \nabla} \equiv_{\infty} A^{\nabla}$; indeed, for any ordinal α which is A^{∇} -writable, $A^{\nabla^{(\alpha)} \nabla} \equiv_{\infty} A^{\nabla}$ (see [HL00]).

8 Post's Problem for Supertasks

Post's problem is the question in classical computability theory of whether there are any non-decidable semi-decidable degrees strictly below the halting problem, or equivalently, whether there are any intermediate semidecidable degrees between 0 and the Turing jump 0'. This question has a natural supertask analogue:

Supertask Post's Problem 9 Are there any intermediate semi-decidable supertask degrees between 0 and the supertask jump 0^{\heartsuit} ?

The answer is delicately mixed. On the one hand, in the context of degrees in the real numbers, we have a negative answer. This contrasts sharply with the classical theory.

Theorem 10. There are no reals z such that $0 <_{\infty} z <_{\infty} 0^{\nabla}$.

Proof. Suppose that $0 \leq_{\infty} z \leq_{\infty} 0^{\nabla}$. So z is the output of program p using 0^{∇} as an oracle. Consider the algorithm which computes approximations to 0^{∇} , and uses program p with these approximations in an attempt to produce z. If one of the proper approximations to 0^{∇} can successfully produce z, then z is writable and $0 \equiv_{\infty} z$. Conversely, if none of the proper approximations can produce z, then on input z we can recognize 0^{∇} as the *true approximation*, the first approximation able to produce z. So $z \equiv_{\infty} 0^{\nabla}$.

On the other hand, when it comes to *sets* of reals, we have an affirmative answer.

Theorem 11. There are semi-decidable sets of reals A with $0 <_{\infty} A <_{\infty} 0^{\nabla}$. Indeed, there are incomparable semi-decidable sets $A \perp B$.

Please consult [HL00] for the proof. Let me mention here, though, that the basic idea of the argument is to generalize the Friedburg-Munchnik priority argument to the supertask context, much as Sacks' did for α recursion theory. Building A and B in stages, we attempt to meet the requirements

$$\varphi_p^B \neq A$$
 and $\varphi_p^A \neq B$

by adding writable reals to A and B that have not yet appeared on the higher priority computations. One technical fact to make this idea work is that for any clockable ordinal α , there are many writable reals not appearing during the course of any supertask computation of length α . Thus, we can find a supply of new writable reals to add to A and B in order to satisfy the later requirements, without injuring the witnessing computations of earlier higher-priority requirements.

9 Other Models of Infinitary Computation

Let me briefly compare the infinite time Turing machine model of supertask computation with some other well-known models.

The Blum-Shub-Smale machines (see [BSS89]) were the original inspiration for infinite time Turing machines.³ Programs and computations for BSS machines are finite, but the basic units of computation are full precision real numbers. They are in essence finite state register machines, where the registers each hold a real number. The primary purpose of introducing the BSS machines was to provide a theoretical foundation for analyzing computational algorithms using the concepts of real analysis rather than arithmetic. The machines allow one to analyze the dynamical features, for example, of actual algorithms in numerical analysis, such as Newton's method, and illuminate questions of stability and convergence for such algorithms. The classical approach to these problems, using the Turing machine model with ever greater decimal approximations, forces one into the realm of finite combinatorics, where one becomes lost in a

³ Jeff Kidder and I heard Lenore Blum's lectures for the Berkeley Logic Colloquium in 1989, and had the idea to generalize the Turing machine concept in a different direction: to infinite time rather than infinite precision.

jumble of discrete approximation error analysis, when one would rather fly smoothly above it in the heaven of differential equations.

In another direction, the theory of higher recursion provides a model of infinitary computability by setting a very general theoretical context for recursion on infinite objects, and one should expect many parallels between it and the theory of infinite time Turing machines. The anonymous referee of [HL] and Philip Welch have pointed out, for example, that the infinitary priority argument [HL, Theorem 4.1], stated as Theorem 11 above, parallels Sacks' version of the Friedburg-Munchnik proof for α -recursion [Sac90], specifically when α is λ , the supremum of the clockable ordinals. One can identify the writable reals in our argument with the ordinal stages at which they appear and get Sacks' sets, and conversely, Sacks' could have written out codes for those stages and gotten our sets. This identification reveals that the \leq_{∞} -degree structure of sets of writable reals below 0^{∇} is exactly that of the λ -degrees. Accordingly, one can obtain not only the answer to Post's problem, but all the theorems from λ -recursion theory for this class of degrees, such as the Shore Density Theorem, etc., for free. It will be very interesting to see if these ideas will allow one to prove the theorems in the general case of all degrees.

Lastly, let me mention quantum Turing machines, if only because I am often asked about them in connection with infinite time Turing machines. Quantum Turing machines are like classical Turing machines, except that the configuration of the machine at any given stage is a superposition of classical configurations; the different components of these superpositions, like the wave functions of quantum mechanics, may constructively or destructively interfere with one another as the computation proceeds. By means of clever quantum algorithms, one can effectively carry out parallel computation in these different components, constructively interfering their output to assemble the information into a final answer. In this way, quantum Turing machines allow for an exponential increase in the speed of computation of many important functions. But because quantum Turing machines, at the end of the day, are simulable by classical Turing machines, they do not introduce new decidable sets or new computable functions. And so while quantum Turing machines are without a doubt extremely important in matters of computational feasibility, they do not really provide a model of infinitary computability. Infinite time Turing machines are simply much more powerful than quantum Turing machines.

10 Questions for the Future

I close this article by asking the open-ended question:

Question 2. What is the structure of infinite time Turing degrees? To what extent do its properties mirror or differ from the classical structure?

This question really stands for the dozens of specific open questions that one might ask: does the Sacks Density Theorem, for example, hold in the supertask context for arbitrary sets of reals? The field is wide open.

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